THE WORKS OF ARCHIMEDES

Edited by T.L. Heath

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F. HULTSCH, *Heronis Alexandrini geometricorum et stereometricorum reliquiae.* (Berlin, 1864.)

F. HULTSCH, *Pappi Alexandrini collectionis quae supersunt.* (Berlin, 1876–8.)

GINO LORIA, *Il periodo aureo della geometria greca.* (Modena, 1895.)

MAXIMILIEN MARIE, *Histoire des sciences mathématiques et physiques,* Tome I. (Paris, 1883.)

J. H. T. MÜLLER, *Beiträge zur Terminologie der griechischen Mathematiker.* (Leipzig, 1860.)

G. H. F. NESSELMANN, *Die Algebra der Griechen.* (Berlin, 1842.)

F. SUSEMIHL, *Geschichte der griechischen Litteratur in der Alexandrinerzeit,* Band I. (Leipzig, 1891.)

P. TANNERY, *La Géométrie grecque,* Première partie, *Histoire générale de la Géométrie élémentaire.* (Paris, 1887.)

H. G. ZEUTHEN, *Die Lehre von den Kegelschnitten im Altertum.* (Copenhagen, 1886.)

H. G. ZEUTHEN, *Geschichte der Mathematik im Altertum und Mittelalter.* (Copenhagen, 1896.)

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THE WORKS OF ARCHIMEDES.

INTRODUCTION.

CHAPTER I.

ARCHIMEDES.

A LIFE of Archimedes was written by one Heracleides*, but this biography has not survived, and such particulars as are known have to be collected from many various sources†. According to Tzetzes‡ he died at the age of 75, and, as he perished in the sack of Syracuse (B.C. 212), it follows that he was probably born about 287 B.C. He was the son of Pheidias the astronomer§, and was on intimate terms with, if not related to, king Hieron and his

* Eutocius mentions this work in his commentary on Archimedes' *Measurement of the circle*, ὥς φησιν Ἡρακλείδης ἐν τῷ Ἀρχιμήδους βίῳ. He alludes to it again in his commentary on Apollonius' *Conics* (ed. Heiberg, Vol. II. p. 168), where, however, the name is wrongly given as Ἡράκλειος. This Heracleides is perhaps the same as the Heracleides mentioned by Archimedes himself in the preface to his book *On Spirals*.

† An exhaustive collection of the materials is given in Heiberg's *Quaestiones Archimedeae* (1879). The preface to Torelli's edition also gives the main points, and the same work (pp. 363—370) quotes at length most of the original references to the mechanical inventions of Archimedes. Further, the article *Archimedes* (by Hultsch) in Pauly-Wissowa's *Real-Encyclopädie der classischen Altertumswissenschaften* gives an entirely admirable summary of all the available information. See also Susemihl's *Geschichte der griechischen Litteratur in der Alexandrinerzeit*, I. pp. 723—733.

‡ Tzetzes, *Chiliad.*, II. 35, 105.

§ Pheidias is mentioned in the *Sand-reckoner* of Archimedes, τῶν προτέρων ἀστρολόγων Εὐδόξου...Φειδία δὲ τοῦ ἁμοῦ πατρὸς (the last words being the correction of Blass for τοῦ Ἀκούπατρος, the reading of the text). Cf. Schol. Clark. in Gregor. Nazianz. Or. 34, p. 355 a Morel. Φειδίας τὸ μὲν γένος ἦν Συρακόσιος ἀστρολόγος ὁ Ἀρχιμήδους πατήρ.

son Gelon. It appears from a passage of Diodorus* that he spent a considerable time at Alexandria, where it may be inferred that he studied with the successors of Euclid. It may have been at Alexandria that he made the acquaintance of Conon of Samos (for whom he had the highest regard both as a mathematician and as a personal friend) and of Eratosthenes. To the former he was in the habit of communicating his discoveries before their publication, and it is to the latter that the famous Cattle-problem purports to have been sent. Another friend, to whom he dedicated several of his works, was Dositheus of Pelusium, a pupil of Conon, presumably at Alexandria though at a date subsequent to Archimedes' sojourn there.

After his return to Syracuse he lived a life entirely devoted to mathematical research. Incidentally he made himself famous by a variety of ingenious mechanical inventions. These things were however merely the "diversions of geometry at play †," and he attached no importance to them. In the words of Plutarch, "he possessed so high a spirit, so profound a soul, and such treasures of scientific knowledge that, though these inventions had obtained for him the renown of more than human sagacity, he yet would not deign to leave behind him any written work on such subjects, but, regarding as ignoble and sordid the business of mechanics and every sort of art which is directed to use and profit, he placed his whole ambition in those speculations in whose beauty and subtlety there is no admixture of the common needs of life ‡." In fact he wrote only one such mechanical book, *On Sphere-making* §, to which allusion will be made later.

Some of his mechanical inventions were used with great effect against the Romans during the siege of Syracuse. Thus he contrived

* Diodorus v. 37, 3, οὓς [τοὺς κοχλίας] Ἀρχιμήδης ὁ Συρακόσιος εὗρεν, ὅτε παρέβαλεν εἰς Αἴγυπτον.

† Plutarch, *Marcellus*, 14.

‡ *ibid.* 17.

§ Pappus VIII. p. 1026 (ed. Hultsch). Κάρπος δέ πού φησιν ὁ Ἀντιοχεὺς Ἀρχιμήδη τὸν Συρακόσιον ἓν μόνον βιβλίον συντεταχέναι μηχανικὸν τὸ κατὰ τὴν σφαιροποιΐαν, τῶν δὲ ἄλλων οὐδὲν ἠξιωκέναι συντάξαι. καίτοι παρὰ τοῖς πολλοῖς ἐπὶ μηχανικῇ δοξασθεὶς καὶ μεγαλοφυής τις γενόμενος ὁ θαυμαστὸς ἐκεῖνος, ὥστε διαμεῖναι παρὰ πᾶσιν ἀνθρώποις ὑπερβαλλόντως ὑμνούμενος, τῶν τε προηγουμένων γεωμετρικῆς καὶ ἀριθμητικῆς ἐχομένων θεωρίας τὰ βραχύτατα δοκοῦντα εἶναι σπουδαίως συνέγραφεν· ὃς φαίνεται τὰς εἰρημένας ἐπιστήμας οὕτως ἀγαπήσας ὡς μηδὲν ἔξωθεν ὑπομένειν αὐταῖς ἐπεισάγειν.

catapults so ingeniously constructed as to be equally serviceable at long or short ranges, machines for discharging showers of missiles through holes made in the walls, and others consisting of long moveable poles projecting beyond the walls which either dropped heavy weights upon the enemy's ships, or grappled the prows by means of an iron hand or a beak like that of a crane, then lifted them into the air and let them fall again*. Marcellus is said to have derided his own engineers and artificers with the words, "Shall we not make an end of fighting against this geometrical Briareus who, sitting at ease by the sea, plays pitch and toss with our ships to our confusion, and by the multitude of missiles that he hurls at us outdoes the hundred-handed giants of mythology?†"; but the exhortation had no effect, the Romans being in such abject terror that "if they did but see a piece of rope or wood projecting above the wall, they would cry 'there it is again,' declaring that Archimedes was setting some engine in motion against them, and would turn their backs and run away, insomuch that Marcellus desisted from all conflicts and assaults, putting all his hope in a long siege ‡."

If we are rightly informed, Archimedes died, as he had lived, absorbed in mathematical contemplation. The accounts of the exact circumstances of his death differ in some details. Thus Livy says simply that, amid the scenes of confusion that followed the capture of Syracuse, he was found intent on some figures which he had drawn in the dust, and was killed by a soldier who did not know who he was §. Plutarch gives more than one version in the following passage. "Marcellus was most of all afflicted at the death of Archimedes; for, as fate would have it, he was intent on working out some problem with a diagram and, having fixed his mind and his eyes alike on his investigation, he never noticed the incursion of the Romans nor the capture of the city. And when a soldier came up to him suddenly and bade him follow to

* Polybius, *Hist.* VIII. 7—8; Livy XXIV. 34; Plutarch, *Marcellus*, 15—17.

† Plutarch, *Marcellus*, 17.

‡ *ibid.*

§ Livy XXV. 31. Cum multa irae, multa auaritiae foeda exempla ederentur, Archimedem memoriae proditum est in tanto tumultu, quantum pauor captae urbis in discursu diripientium militum ciere poterat, intentum formis, quas in puluere descripserat, ab ignaro milite quis esset interfectum; aegre id Marcellum tulisse sepulturaeque curam habitam, et propinquis etiam inquisitis honori praesidioque nomen ac memoriam eius fuisse.

Marcellus, he refused to do so until he had worked out his problem to a demonstration; whereat the soldier was so enraged that he drew his sword and slew him. Others say that the Roman ran up to him with a drawn sword offering to kill him; and, when Archimedes saw him, he begged him earnestly to wait a short time in order that he might not leave his problem incomplete and unsolved, but the other took no notice and killed him. Again there is a third account to the effect that, as he was carrying to Marcellus some of his mathematical instruments, sundials, spheres, and angles adjusted to the apparent size of the sun to the sight, some soldiers met him and, being under the impression that he carried gold in the vessel, slew him*." The most picturesque version of the story is perhaps that which represents him as saying to a Roman soldier who came too close, "Stand away, fellow, from my diagram," whereat the man was so enraged that he killed him†. The addition made to this story by Zonaras, representing him as saying παρὰ κεφαλὰν καὶ μὴ παρὰ γραμμάν, while it no doubt recalls the second version given by Plutarch, is perhaps the most far-fetched of the touches put to the picture by later hands.

Archimedes is said to have requested his friends and relatives to place upon his tomb a representation of a cylinder circumscribing a sphere within it, together with an inscription giving the ratio which the cylinder bears to the sphere‡; from which we may infer that he himself regarded the discovery of this ratio [*On the Sphere and Cylinder*, I. 33, 34] as his greatest achievement. Cicero, when quaestor in Sicily, found the tomb in a neglected state and restored it§.

Beyond the above particulars of the life of Archimedes, we have nothing left except a number of stories, which, though perhaps not literally accurate, yet help us to a conception of the personality of the most original mathematician of antiquity which we would not willingly have altered. Thus, in illustration of his entire preoccupation by his abstract studies, we are told that he would forget all about his food and such necessities of life, and would be drawing geometrical figures in the ashes of the fire, or, when

* Plutarch, *Marcellus*, 19.

† Tzetzes, *Chil.* II. 35, 135; Zonaras IX. 5.

‡ Plutarch, *Marcellus*, 17 *ad fin.*

§ Cicero, *Tusc.* V. 64 sq.

anointing himself, in the oil on his body*. Of the same kind is the well-known story that, when he discovered in a bath the solution of the question referred to him by Hieron as to whether a certain crown supposed to have been made of gold did not in reality contain a certain proportion of silver, he ran naked through the street to his home shouting εὕρηκα, εὕρηκα†.

According to Pappus‡ it was in connexion with his discovery of the solution of the problem *To move a given weight by a given force* that Archimedes uttered the famous saying, "Give me a place to stand on, and I can move the earth (δός μοι ποῦ στῶ καὶ κινῶ τὴν γῆν)." Plutarch represents him as declaring to Hieron that any given weight could be moved by a given force, and boasting, in reliance on the cogency of his demonstration, that, if he were given another earth, he would cross over to it and move this one. "And when Hieron was struck with amazement and asked him to reduce the problem to practice and to give an illustration of some great weight moved by a small force, he fixed upon a ship of burden with three masts from the king's arsenal which had only been drawn up with great labour and many men; and loading her with many passengers and a full freight, sitting himself the while far off, with no great endeavour but only holding the end of a compound pulley (πολύσπαστος) quietly in his hand and pulling at it, he drew the ship along smoothly and safely as if she were moving through the sea§." According to Proclus the ship was one which Hieron had had made to send to king Ptolemy, and, when all the Syracusans with their combined strength were unable to launch it, Archimedes contrived a mechanical device which enabled Hieron to move it by himself, insomuch that the latter declared that "from that day forth Archimedes was to be believed in everything that he might say‖." While however it is thus established that Archimedes invented some mechanical contrivance for moving a large ship and thus gave a practical illustration of his thesis, it is not certain whether the machine used was simply a compound

* Plutarch, *Marcellus*, 17.

† Vitruvius, *Architect.* IX. 3. For an explanation of the manner in which Archimedes probably solved this problem, see the note following *On floating bodies*, I. 7 (p. 259 sq.).

‡ Pappus VIII. p. 1060.

§ Plutarch, *Marcellus*, 14.

‖ Proclus, *Comm. on Eucl.* I., p. 63 (ed. Friedlein).

pulley (πολύσπαστος) as stated by Plutarch; for Athenaeus*, in describing the same incident, says that a *helix* was used. This term must be supposed to refer to a machine similar to the κοχλίας described by Pappus, in which a cog-wheel with oblique teeth moves on a cylindrical helix turned by a handle†. Pappus, however, describes it in connexion with the βαρουλκός of Heron, and, while he distinctly refers to Heron as his authority, he gives no hint that Archimedes invented either the βαρουλκός or the particular κοχλίας; on the other hand, the πολύσπαστος is mentioned by Galen‡, and the τρίσπαστος (triple pulley) by Oribasius§, as one of the inventions of Archimedes, the τρίσπαστος being so called either from its having three wheels (Vitruvius) or three ropes (Oribasius). Nevertheless, it may well be that though the ship could easily be kept in motion, when once started, by the τρίσπαστος or πολύσπαστος, Archimedes was obliged to use an appliance similar to the κοχλίας to give the first impulse.

The name of yet another instrument appears in connexion with the phrase about moving the earth. Tzetzes' version is, "Give me a place to stand on (πᾶ βῶ), and I will move the whole earth with a χαριστίων‖"; but, as in another passage¶ he uses the word τρίσπαστος, it may be assumed that the two words represented one and the same thing**.

It will be convenient to mention in this place the other mechanical inventions of Archimedes. The best known is the

* Athenaeus v. 207 a–b, κατασκευάσας γὰρ ἕλικα τὸ τηλικοῦτον σκάφος εἰς τὴν θάλασσαν κατήγαγε· πρῶτος δ' Ἀρχιμήδης εὗρε τὴν τῆς ἕλικος κατασκευήν. To the same effect is the statement of Eustathius *ad Il.* III. p. 114 (ed. Stallb.) λέγεται δὲ ἕλιξ καί τι μηχανῆς εἶδος, ὃ πρῶτος εὑρὼν ὁ Ἀρχιμήδης εὐδοκίμησέ, φασι, δι' αὐτοῦ.

† Pappus VIII. pp. 1066, 1108 sq.

‡ Galen, *in Hippocr. De artic.*, IV. 47 (=XVIII. p. 747, ed. Kühn).

§ Oribasius, *Coll. med.*, XLIX. 22 (IV. p. 407, ed. Bussemaker), Ἀπελλίδους ἢ Ἀρχιμήδους τρίσπαστον, described in the same passage as having been invented πρὸς τὰς τῶν πλοίων καθολκάς.

‖ Tzetzes, *Chil.* II. 130.

¶ *Ibid.*, III. 61, ὁ γῆν ἀνασπῶν μηχανῇ τῇ τρισπάστῳ βοῶν· ὅπα βῶ καὶ σαλεύσω τὴν χθόνα.

** Heiberg compares Simplicius, *Comm. in Aristot. Phys.* (ed. Diels, p. 1110, l. 2), ταύτῃ δὲ τῇ ἀναλογίᾳ τοῦ κινοῦντος καὶ τοῦ κινουμένου καὶ τοῦ διαστήματος τὸ σταθμιστικὸν ὄργανον τὸν καλούμενον χαριστίωνα συστήσας ὁ Ἀρχιμήδης ὡς μέχρι παντὸς τῆς ἀναλογίας προχωρούσης ἐκόμπασεν ἐκεῖνο τὸ πᾶ βῶ καὶ κινῶ τὰν γᾶν.

water-screw* (also called κοχλίας) which was apparently invented by him in Egypt, for the purpose of irrigating fields. It was also used for pumping water out of mines or from the hold of ships.

Another invention was that of a sphere constructed so as to imitate the motions of the sun, the moon, and the five planets in the heavens. Cicero actually saw this contrivance and gives a description of it†, stating that it represented the periods of the moon and the apparent motion of the sun with such accuracy that it would even (over a short period) show the eclipses of the sun and moon. Hultsch conjectures that it was moved by water‡. We know, as above stated, from Pappus that Archimedes wrote a book on the construction of such a sphere (περὶ σφαιροποιίας), and Pappus speaks in one place of "those who understand the making of spheres and produce a model of the heavens by means of the regular circular motion of water." In any case it is certain that Archimedes was much occupied with astronomy. Livy calls him "unicus spectator caeli siderumque." Hipparchus says§, "From these observations it is clear that the differences in the years are altogether small, but, as to the solstices, I almost think (οὐκ ἀπελπίζω) that both I and Archimedes have erred to the extent of a quarter of a day both in the observation and in the deduction therefrom." It appears therefore that Archimedes had considered the question of the length of the year, as Ammianus also states‖. Macrobius says that he discovered the distances of the planets¶. Archimedes himself describes in the *Sand-reckoner* the apparatus by which he measured the apparent diameter of the sun, or the angle subtended by it at the eye.

The story that he set the Roman ships on fire by an arrangement of burning-glasses or concave mirrors is not found in any

* Diodorus I. 34, V. 37; Vitruvius X. 16 (11); Philo III. p. 330 (ed. Pfeiffer); Strabo XVII. p. 807; Athenaeus V. 208 f.

† Cicero, *De rep.*, I. 21–22; *Tusc.*, I. 63; *De nat. deor.*, II. 88. Cf. Ovid, *Fasti*, VI. 277; Lactantius, *Instit.*, II. 5, 18; Martianus Capella, II. 212, VI. 583 sq.; Claudian, *Epigr.* 18; Sextus Empiricus, p. 416 (ed. Bekker).

‡ *Zeitschrift f. Math. u. Physik* (*hist. litt. Abth.*), XXII. (1877), 106 sq.

§ Ptolemy, σύνταξις, I. p. 153.

‖ Ammianus Marcell., XXVI. i. 8.

¶ Macrobius, *in Somn. Scip.*, II. 3.

authority earlier than Lucian*; and the so-called *loculus Archimedius*, which was a sort of puzzle made of 14 pieces of ivory of different shapes cut out of a square, cannot be supposed to be his invention, the explanation of the name being perhaps that it was only a method of expressing that the puzzle was cleverly made, in the same way as the πρόβλημα Ἀρχιμήδειον came to be simply a proverbial expression for something very difficult†.

* The same story is told of Proclus in Zonaras XIV. 3. For the other references on the subject see Heiberg's *Quaestiones Archimedeae*, pp. 39–41.

† Cf. also Tzetzes, *Chil.* XII. 270, τῶν Ἀρχιμήδους μηχανῶν χρείαν ἔχω.

CHAPTER II.

MANUSCRIPTS AND PRINCIPAL EDITIONS—ORDER OF COMPOSITION—DIALECT—LOST WORKS.

THE sources of the text and versions are very fully described by Heiberg in the Prolegomena to Vol. III. of his edition of Archimedes, where the editor supplements and to some extent amends what he had previously written on the same subject in his dissertation entitled *Quaestiones Archimedeae* (1879). It will therefore suffice here to state briefly the main points of the discussion.

The MSS. of the best class all had a common origin in a MS. which, so far as is known, is no longer extant. It is described in one of the copies made from it (to be mentioned later and dating from some time between A.D. 1499 and 1531) as 'most ancient' (παλαιοτάτου), and all the evidence goes to show that it was written as early as the 9th or 10th century. At one time it was in the possession of George Valla, who taught at Venice between the years 1486 and 1499; and many important inferences with regard to its readings can be drawn from some translations of parts of Archimedes and Eutocius made by Valla himself and published in his book entitled *de expetendis et fugiendis rebus* (Venice, 1501). It appears to have been carefully copied from an original belonging to some one well versed in mathematics, and it contained figures drawn for the most part with great care and accuracy, but there was considerable confusion between the letters in the figures and those in the text. This MS., after the death of Valla in 1499, became the property of Albertus Pius Carpensis (Alberto Pio, prince of Carpi). Part of his library passed through various hands and ultimately reached the Vatican; but the fate of the Valla MS. appears to have been different, for we hear of its being in the possession of Cardinal Rodolphus Pius (Rodolfo Pio), a nephew of Albertus, in 1544, after which it seems to have disappeared.

If then $\triangle PQq = A$, $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, and so on, until we arrive at an area X such that X is less than the difference between K and the segment, we have

$$A + B + C + \ldots + X + \tfrac{1}{3}X = \tfrac{4}{3}A \qquad \text{[Prop. 23]}$$
$$= K.$$

Now, since K exceeds $A + B + C + \ldots + X$ by an area less than X, and the area of the segment by an area greater than X, it follows that

$$A + B + C + \ldots + X > \text{(the segment)};$$

which is impossible, by Prop. 22 above.

Hence the segment is not less than K.

Thus, since the segment is neither greater nor less than K,

$$(\text{area of segment } PQq) = K = \tfrac{4}{3}\triangle PQq.$$

ON FLOATING BODIES.

BOOK I.

Postulate 1.

"Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more; and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in anything and compressed by anything else."

Proposition 1.

If a surface be cut by a plane always passing through a certain point, and if the section be always a circumference [of a circle] whose centre is the aforesaid point, the surface is that of a sphere.

For, if not, there will be some two lines drawn from the point to the surface which are not equal.

Suppose O to be the fixed point, and A, B to be two points on the surface such that OA, OB are unequal. Let the surface be cut by a plane passing through OA, OB. Then the section is, by hypothesis, a circle whose centre is O.

Thus $OA = OB$; which is contrary to the assumption. Therefore the surface cannot but be a sphere.

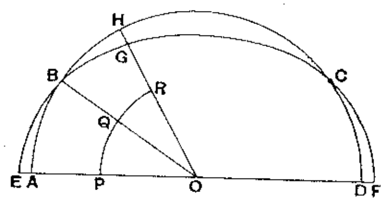
Proposition 2.

The surface of any fluid at rest is the surface of a sphere whose centre is the same as that of the earth.

Suppose the surface of the fluid cut by a plane through O, the centre of the earth, in the curve $ABCD$.

$ABCD$ shall be the circumference of a circle.

For, if not, some of the lines drawn from O to the curve will be unequal. Take one of them, OB, such that OB is greater than some of the lines from O to the curve and less than others. Draw a circle with OB as radius. Let it be EBF, which will therefore fall partly within and partly without the surface of the fluid.



Draw OGH making with OB an angle equal to the angle EOB, and meeting the surface in H and the circle in G. Draw also in the plane an arc of a circle PQR with centre O and within the fluid.

Then the parts of the fluid along PQR are uniform and continuous, and the part PQ is compressed by the part between it and AB, while the part QR is compressed by the part between QR and BH. Therefore the parts along PQ, QR will be unequally compressed, and the part which is compressed the less will be set in motion by that which is compressed the more.

Therefore there will not be rest; which is contrary to the hypothesis.

Hence the section of the surface will be the circumference of a circle whose centre is O; and so will all other sections by planes through O.

Therefore the surface is that of a sphere with centre O.

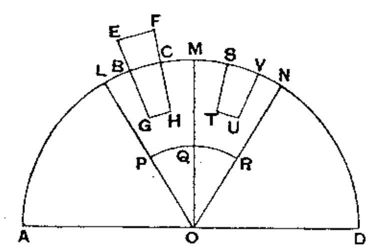
Proposition 3.

Of solids those which, size for size, are of equal weight with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface but do not sink lower.

If possible, let a certain solid $EFHG$ of equal weight, volume for volume, with the fluid remain immersed in it so that part of it, $EBCF$, projects above the surface.

Draw through O, the centre of the earth, and through the solid a plane cutting the surface of the fluid in the circle $ABCD$.

Conceive a pyramid with vertex O and base a parallelogram at the surface of the fluid, such that it includes the immersed portion of the solid. Let this pyramid be cut by the plane of $ABCD$ in OL, OM. Also let a sphere within the fluid and below GH be described with centre O, and let the plane of $ABCD$ cut this sphere in PQR.



Conceive also another pyramid in the fluid with vertex O, continuous with the former pyramid and equal and similar to it. Let the pyramid so described be cut in OM, ON by the plane of $ABCD$.

Lastly, let $STUV$ be a part of the fluid within the second pyramid equal and similar to the part $BGHC$ of the solid, and let SV be at the surface of the fluid.

Then the pressures on PQ, QR are unequal, that on PQ being the greater. Hence the part at QR will be set in motion

by that at PQ, and the fluid will not be at rest; which is contrary to the hypothesis.

Therefore the solid will not stand out above the surface.

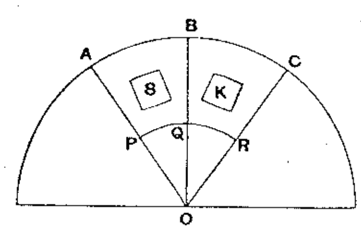
Nor will it sink further, because all the parts of the fluid will be under the same pressure.

Proposition 4.

A solid lighter than a fluid will, if immersed in it, not be completely submerged, but part of it will project above the surface.

In this case, after the manner of the previous proposition, we assume the solid, if possible, to be completely submerged and the fluid to be at rest in that position, and we conceive (1) a pyramid with its vertex at O, the centre of the earth, including the solid, (2) another pyramid continuous with the former and equal and similar to it, with the same vertex O, (3) a portion of the fluid within this latter pyramid equal to the immersed solid in the other pyramid, (4) a sphere with centre O whose surface is below the immersed solid and the part of the fluid in the second pyramid corresponding thereto. We suppose a plane to be drawn through the centre O cutting the surface of the fluid in the circle ABC, the solid in S, the first pyramid in OA, OB, the second pyramid in OB, OC, the portion of the fluid in the second pyramid in K, and the inner sphere in PQR.

Then the pressures on the parts of the fluid at PQ, QR are unequal, since S is lighter than K. Hence there will not be rest; which is contrary to the hypothesis.

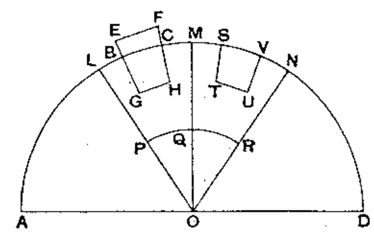


Therefore the solid S cannot, in a condition of rest, be completely submerged.

Proposition 5.

Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.

For let the solid be $EGHF$, and let $BGHC$ be the portion of it immersed when the fluid is at rest. As in Prop. 3, conceive a pyramid with vertex O including the solid, and another pyramid with the same vertex continuous with the former and equal and similar to it. Suppose a portion of the fluid $STUV$ at the base of the second pyramid to be equal and similar to the immersed portion of the solid; and let the construction be the same as in Prop. 3.



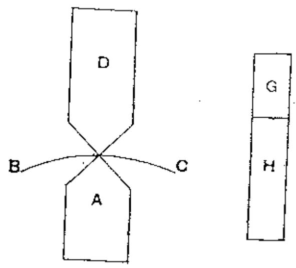
Then, since the pressure on the parts of the fluid at PQ, QR must be equal in order that the fluid may be at rest, it follows that the weight of the portion $STUV$ of the fluid must be equal to the weight of the solid $EGHF$. And the former is equal to the weight of the fluid displaced by the immersed portion of the solid $BGHC$.

Proposition 6.

If a solid lighter than a fluid be forcibly immersed in it, the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced.

For let A be completely immersed in the fluid, and let G represent the weight of A, and $(G + H)$ the weight of an equal volume of the fluid. Take a solid D, whose weight is H

and add it to A. Then the weight of $(A + D)$ is less than that of an equal volume of the fluid; and, if $(A + D)$ is immersed in the fluid, it will project so that its weight will be equal to the weight of the fluid displaced. But its weight is $(G + H)$.



Therefore the weight of the fluid displaced is $(G + H)$, and hence the volume of the fluid displaced is the volume of the solid A. There will accordingly be rest with A immersed and D projecting.

Thus the weight of D balances the upward force exerted by the fluid on A, and therefore the latter force is equal to H, which is the difference between the weight of A and the weight of the fluid which A displaces.

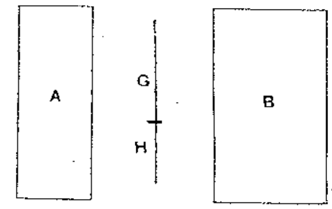
Proposition 7.

A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.

(1) The first part of the proposition is obvious, since the part of the fluid under the solid will be under greater pressure, and therefore the other parts will give way until the solid reaches the bottom.

(2) Let A be a solid heavier than the same volume of the fluid, and let $(G + H)$ represent its weight, while G represents the weight of the same volume of the fluid.

Take a solid B lighter than the same volume of the fluid, and such that the weight of B is G, while the weight of the same volume of the fluid is $(G + H)$.



Let A and B be now combined into one solid and immersed. Then, since $(A + B)$ will be of the same weight as the same volume of fluid, both weights being equal to $(G + H) + G$, it follows that $(A + B)$ will remain stationary in the fluid.

Therefore the force which causes A by itself to sink must be equal to the upward force exerted by the fluid on B by itself. This latter is equal to the difference between $(G + H)$ and G [Prop. 6]. Hence A is depressed by a force equal to H, i.e. its weight in the fluid is H, or the difference between $(G + H)$ and G.

[This proposition may, I think, safely be regarded as decisive of the question how Archimedes determined the proportions of gold and silver contained in the famous crown (cf. Introduction, Chapter I.). The proposition suggests in fact the following method.

Let W represent the weight of the crown, w_1 and w_2 the weights of the gold and silver in it respectively, so that $W = w_1 + w_2$.

(1) Take a weight W of pure gold and weigh it in a fluid. The apparent loss of weight is then equal to the weight of the fluid displaced. If F_1 denote this weight, F_1 is thus known as the result of the operation of weighing.

It follows that the weight of fluid displaced by a weight w_1 of gold is $\frac{w_1}{W} \cdot F_1$.

(2) Take a weight W of pure silver and perform the same operation. If F_2 be the loss of weight when the silver is weighed in the fluid, we find in like manner that the weight of fluid displaced by w_2 is $\frac{w_2}{W} \cdot F_2$.

(3) Lastly, weigh the crown itself in the fluid, and let F be the loss of weight. Therefore the weight of fluid displaced by the crown is F.

It follows that $$\frac{w_1}{W} \cdot F_1 + \frac{w_2}{W} \cdot F_2 = F,$$

or $$w_1 F_1 + w_2 F_2 = (w_1 + w_2) F,$$

whence $$\frac{w_1}{w_2} = \frac{F_2 - F}{F - F_1}.$$

This procedure corresponds pretty closely to that described in the poem *de ponderibus et mensuris* (written probably about 500 A.D.)* purporting to explain Archimedes' method. According to the author of this poem, we first take two equal weights of pure gold and pure silver respectively and weigh them against each other when both immersed in water; this gives the relation between their weights in water and therefore between their loss of weight in water. Next we take the mixture of gold and silver and an equal weight of pure silver and weigh them against each other in water in the same manner.

The other version of the method used by Archimedes is that given by Vitruvius†, according to which he measured successively the *volumes* of fluid displaced by three equal weights, (1) the crown, (2) the same weight of gold, (3) the same weight of silver, respectively. Thus, if as before the weight of the crown is W, and it contains weights w_1 and w_2 of gold and silver respectively,

(1) the crown displaces a certain quantity of fluid, V say.

(2) the weight W of gold displaces a certain volume of fluid, V_1 say; therefore a weight w_1 of gold displaces a volume $\frac{w_1}{W} \cdot V_1$ of fluid.

(3) the weight W of silver displaces a certain volume of fluid, say V_2; therefore a weight w_2 of silver displaces a volume $\frac{w_2}{W} \cdot V_2$ of fluid.

It follows that $$V = \frac{w_1}{W} \cdot V_1 + \frac{w_2}{W} \cdot V_2,$$

whence, since $$W = w_1 + w_2,$$
$$\frac{w_1}{w_2} = \frac{V_2 - V}{V - V_1};$$

and this ratio is obviously equal to that before obtained, viz. $\frac{F_2 - F}{F - F_1}$.]

* Torelli's *Archimedes*, p. 364; Hultsch, *Metrol. Script.* II. 95 sq., and Prolegomena § 118.

† *De architect.* IX. 3.

Postulate 2.

"Let it be granted that bodies which are forced upwards in a fluid are forced upwards along the perpendicular [to the surface] which passes through their centre of gravity."

Proposition 8.

If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base does not touch the surface, the solid will rest in such a position that its axis is perpendicular to the surface; and, if the solid be forced into such a position that its base touches the fluid on one side and be then set free, it will not remain in that position but will return to the symmetrical position.

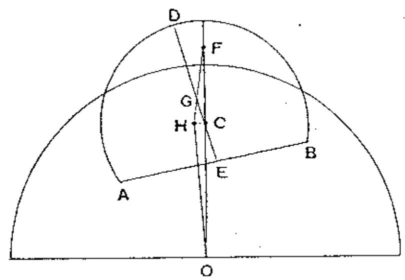
[The proof of this proposition is wanting in the Latin version of Tartaglia. Commandinus supplied a proof of his own in his edition.]

Proposition 9.

If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base is completely below the surface, the solid will rest in such a position that its axis is perpendicular to the surface.

[The proof of this proposition has only survived in a mutilated form. It deals moreover with only one case out of three which are distinguished at the beginning, viz. that in which the segment is greater than a hemisphere, while figures only are given for the cases where the segment is equal to, or less than, a hemisphere.]

Suppose, first, that the segment is greater than a hemisphere. Let it be cut by a plane through its axis and the centre of the earth; and, if possible, let it be at rest in the position shown in the figure, where AB is the intersection of the plane with the base of the segment, DE its axis, C the centre of the sphere of which the segment is a part, O the centre of the earth.



The centre of gravity of the portion of the segment outside the fluid, as F, lies on OC produced, its axis passing through C.

Let G be the centre of gravity of the segment. Join FG, and produce it to H so that

$FG : GH =$ (volume of immersed portion) : (rest of solid).

Join OH.

Then the weight of the portion of the solid outside the fluid acts along FO, and the pressure of the fluid on the immersed portion along OH, while the weight of the immersed portion acts along HO and is by hypothesis less than the pressure of the fluid acting along OH.

Hence there will not be equilibrium, but the part of the segment towards A will ascend and the part towards B descend, until DE assumes a position perpendicular to the surface of the fluid.

ON FLOATING BODIES.

BOOK II.

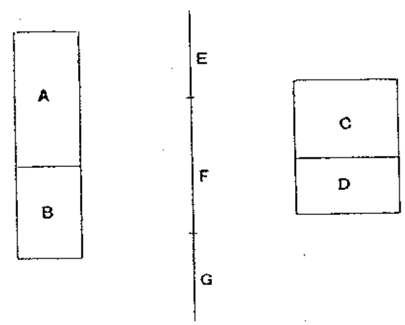
Proposition 1.

If a solid lighter than a fluid be at rest in it, the weight of the solid will be to that of the same volume of the fluid as the immersed portion of the solid is to the whole.

Let $(A + B)$ be the solid, B the portion immersed in the fluid.

Let $(C + D)$ be an equal volume of the fluid, C being equal in volume to A and B to D.

Further suppose the line E to represent the weight of the solid $(A + B)$, $(F + G)$ to represent the weight of $(C + D)$, and G that of D.



Then

weight of $(A + B)$: weight of $(C + D) = E : (F + G)$...(1).

And the weight of $(A+B)$ is equal to the weight of a volume B of the fluid [I. 5], i.e. to the weight of D.

That is to say, $E=G$.

Hence, by (1),

$$\begin{aligned} \text{weight of } (A+B) : \text{weight of } (C+D) &= G : F+G \\ &= D : C+D \\ &= B : A+B. \end{aligned}$$

Proposition 2.

If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{3}{4}p$ (where p is the principal parameter of the generating parabola), and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined to the vertical at any angle, but so that the base of the segment does not touch the surface of the fluid, the segment of the paraboloid will not remain in that position but will return to the position in which its axis is vertical.

Let the axis of the segment of the paraboloid be AN, and through AN draw a plane perpendicular to the surface of the fluid. Let the plane intersect the paraboloid in the parabola BAB', the base of the segment of the paraboloid in BB', and the plane of the surface of the fluid in the chord QQ' of the parabola.

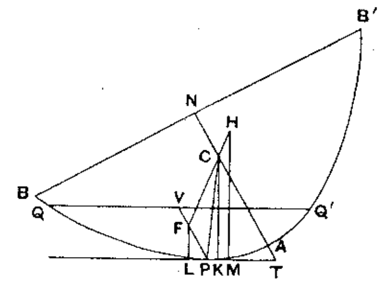
Then, since the axis AN is placed in a position not perpendicular to QQ', BB' will not be parallel to QQ'.

Draw the tangent PT to the parabola which is parallel to QQ', and let P be the point of contact*.

[From P draw PV parallel to AN meeting QQ' in V. Then PV will be a diameter of the parabola, and also the axis of the portion of the paraboloid immersed in the fluid.

* The rest of the proof is wanting in the version of Tartaglia, but is given in brackets as supplied by Commandinus.

Let C be the centre of gravity of the paraboloid BAB', and F that of the portion immersed in the fluid. Join FC and produce it to H so that H is the centre of gravity of the remaining portion of the paraboloid above the surface.



Then, since $AN = \frac{3}{2}AC$*,

and $AN \not> \frac{3}{4}p$,

it follows that $AC \not> \frac{p}{2}$.

Therefore, if CP be joined, the angle CPT is acute†. Hence, if CK be drawn perpendicular to PT, K will fall between P and T. And, if FL, HM be drawn parallel to CK to meet PT, they will each be perpendicular to the surface of the fluid.

Now the force acting on the immersed portion of the segment of the paraboloid will act upwards along LF, while the weight of the portion outside the fluid will act downwards along HM.

Therefore there will not be equilibrium, but the segment

* As the determination of the centre of gravity of a segment of a paraboloid which is here assumed does not appear in any extant work of Archimedes, or in any known work by any other Greek mathematician, it appears probable that it was investigated by Archimedes himself in some treatise now lost.

† The truth of this statement is easily proved from the property of the subnormal. For, if the normal at P meet the axis in G, AG is greater than $\frac{p}{2}$ except in the case where the normal is the normal at the vertex A itself. But the latter case is excluded here because, by hypothesis, AN is not placed vertically. Hence, P being a different point from A, AG is always greater than AC; and, since the angle TPG is right, the angle TPC must be acute.